* The Initial value problem is given by:-

$$\frac{dy}{dx} = f(x, y, z) \quad \text{where } y = y_0 , \; z = z_0 \; \text{ when } x = x_0$$

The nth Approximation of Initial value Problem is

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\, dx$$

$$z_n(x) = z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\, dx$$

Ex-3 $\frac{dy}{dx} = z , \quad \frac{dz}{dx} = x^2 z + x^4 y$

by Picard's Method where $y = 5, \; z = 1$ when $x = 0$

$\Rightarrow$ The Initial value problem is given by -

$$\frac{dy}{dx} = z \quad \frac{dz}{dx} = x^2 z + x^4 y$$

Where $y = 5, \; z = 1$, when $x = 0$

Comparing it with $\frac{dy}{dx} = f(x, y, z)$

where $y = y_0 , \; z = z_0$ when $x = x_0$

$y_0 = 5, \quad z_0 = 1, \quad x_0 = 0$

$$y_n(x) = y_0 + \int_{x_0}^{x} z_{n-1}\, dx$$

$$z_n(x) = z_0 + \int_{x_0}^{x} (x^2 z_{n-1} + x^4 y_{n-1})\, dx$$

Ist Approx :- $n = 1$

$$y_1(x) = 5 + \int_0^x z_0\, dx$$

$$= 5 + \int_0^x 1\, dx$$

$$= 5 + 1x$$

$$= 5 + x$$

$$z_1(x) = 1 + \int_0^x (x^2 z_0 + x^4 y_0)\, dx$$

$$= 1 + \int_0^x (x^2 + 5x^4)\, dx$$

$$= 1 + \left[\frac{x^3}{3} + \frac{5x^5}{5}\right]_0^x$$

$$= 1 + \frac{x^3}{3} + x^5$$

2nd Approx :- $n = 2$

$$y_2(x) = 5 + \int_0^x z_1 \, dx$$

$$= 5 + \int_0^x \left[1 + \frac{x^3}{3} + x^5\right] dx$$

$$= 5 + x + \frac{x^4}{12} + \frac{x^6}{6}$$

$$Z_2(x) = 1 + \int_0^x (x^2 z_1 + x^4 y_1) \, dx$$

$$= 1 + \int_0^x \left(x^2 + \frac{x^5}{3} + x^7 + 5x^4 + x^5\right) dx$$

$$= 1 + \left[\frac{x^3}{3} + \frac{x^6}{18} + \frac{x^8}{8} + \frac{5x^5}{5} + \frac{x^6}{6}\right]_0^x$$

$$= 1 + \frac{x^3}{3} + \frac{x^6}{18} + \frac{x^8}{8} + x^5 + \frac{x^6}{6}$$

$$= 1 + \frac{x^3}{3} + x^5 + \frac{x^6}{18} + \frac{x^6}{6} + \frac{x^8}{8}$$

$$= 1 + \frac{x^3}{3} + x^5 + \frac{x^6 + 3x^6}{18} + \frac{x^8}{8}$$

$$= 1 + \frac{x^3}{3} + x^5 + \frac{4x^6}{18} + \frac{x^8}{8}$$

$$= 1 + \frac{x^3}{3} + x^5 + \frac{2x^6}{9} + \frac{x^8}{8}$$

<u>3rd Approx :-</u> $n = 3$

$$y_3(x) = y_0 + \int_0^x z_2 \, dx$$

$$= 5 + \int_0^x \left[1 + \frac{x^3}{3} + x^5 + \frac{2x^6}{9} + \frac{x^8}{8} \right] dx$$

$$= 5 + \left[x + \frac{x^4}{12} + \frac{x^6}{6} + \frac{2x^7}{63} + \frac{x^9}{72} \right]_0^x$$

$$= 5 + x + \frac{x^4}{12} + \frac{x^6}{6} + \frac{2x^7}{63} + \frac{x^9}{72}$$

$$z_3(x) = z_0 + \int_0^x (x^2 z_2 + x^4 y_2) \, dx$$

$$= 1 + \int_0^x \left[x^2 + \frac{x^5}{3} + x^7 + \frac{2x^8}{9} + \frac{x^{10}}{8} \right] dx$$

$$+ \int_0^x \left[5x^4 + x^5 + \frac{x^8}{12} + \frac{x^{10}}{6} \right] dx$$

$$= 1 + \frac{x^3}{3} + \frac{x^6}{18} + \frac{x^8}{8} + \frac{2x^9}{81} + \frac{x^{11}}{88} + \frac{5x^5}{5} + \frac{x^6}{6}$$

$$+ \frac{x^9}{108} + \frac{x^{11}}{66}$$

$$= 1 + \frac{x^3}{3} + x^5 + \frac{2}{9}x^6 + \frac{1}{8}x^8 + \frac{11}{324}x^9 + \frac{7}{264}x^{11}$$

Ex - 7 $\quad \frac{dy}{dx} = 3e^x + 2y$

where $y = 0$ when $x = 0$

$\rightarrow f(x,y) = 3e^x + 2y$ where $y_0 = 0$ when $x_0 = 0$

Ist Approx :- $n = 1$

$$y_1(x) = y_0 + \int_{x_0}^{x} y_{n-1}\, dx$$

$$y_1(x) = 0 + \int_0^x (3e^x + 2y_0)\, dx$$

$$= \left[3e^x\right]_0^x$$

$$= 3e^x - 3e^0$$

$$= 3(e^x - 1)$$

$$y_2(x) = \int_0^x (3e^x + 2y_1)\, dx$$

$$= \int_0^x (3e^x + 2(3e^x - 3))\, dx$$

$$= \int_0^x (3e^x + 6e^x - 6)\, dx$$

$$= \left[3e^x + 6e^x - 6x\right]_0^x$$

$$= 3e^x + 6e^x - 6x - 3 - 6$$

$$= 9e^x - 6x - 9$$

$$y_3(x) = \int_0^x (3e^x + 2y_2)\, dx$$

$$= \int_0^x (3e^x + 2(9e^x - 6x - 9))\, dx$$

$$= \int_0^x (3e^x + 18e^x - 12x - 18)\, dx$$

$$= \int_0^x \left[21e^x - 12x - 18\right] dx$$

$$= \left[21e^{x} - \frac{12x^{2}}{2} - 18x\right]_{0}^{x}$$

$$= \left[21e^{x} - 6x^{2} - 18x\right]_{0}^{x}$$

$$= 21e^{x} - 6x^{2} - 18x - 21$$